The Suspended Slinky—
A Problem in Static Equilibrium

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The humble Slinky™ spring toy has been known for decades as an invaluable tool for demonstrating wave motions, both longitudinal and transverse. An even simpler situation involving its physical properties is its static configuration when suspended vertically from one end. Such a situation was considered briefly by Heard and Newby[1] in a paper entitled "Behavior of a Soft Spring," but their chief concern was with the vertical oscillations of such a spring with masses attached, and they did not explore the static equilibrium experimentally.

Theory

Assume a Slinky of mass M with a total of N turns. Let its relaxed length (neither stretched nor compressed) be L_0. Suppose that, when suspended under gravity from one end, it has a total length L. The tension developed at any point in the Slinky must be such as to support the weight of the part of the spring toy below that point. In order to analyze this situation, let us measure vertical distance z from the bottom end of the Slinky. Let us also count the number of turns, n, from this bottom point. Let the tension in the Slinky at z be $T(z)$; then we must have:

$$T(z) = \frac{n(z)}{N} Mg \qquad (1)$$

Consider now a very short section of the Slinky between z and $z + \Delta z$ (Fig. 1). The tension $T(z)$ arises from the stretching of this section. Let us suppose that,

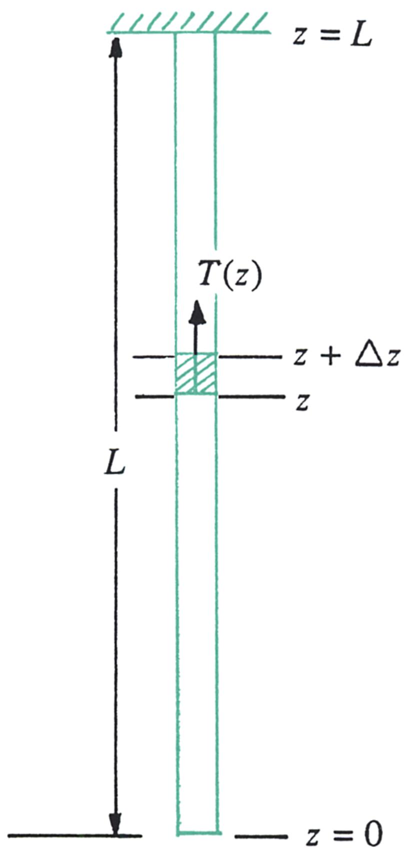

Fig. 1. Diagram of suspended Slinky. The tension *T(z)* must support the weight of the portion of the Slinky below that level.

in the unstretched state, this section has a length Δz_0. If the Slinky can be assumed to conform to Hooke's law, we shall have:

$$T(z) = k\left(\frac{\Delta z - \Delta z_0}{\Delta z_0}\right)$$

where k is a constant with the dimensions of force. (Note that it is not the same as a spring constant—measured for example in N/m—for which the symbol k is also often used.) Rearranging the previous equation, we have:

$$\Delta z = \Delta z_0 \left[1 + \frac{T(z)}{k}\right]$$

We also need to satisfy the condition that the *change* of tension between z and $z + \Delta z$ be such as to support the weight of this short section of the toy. If we denote the mass per unit length of the unstretched Slinky as μ_0 ($= M/L_0$), the mass of this section is equal to $\mu_0 \Delta z_0$ and so we have:

$$\Delta T = g\,\mu_0\,\Delta z_0 = \frac{g\,\mu_0\,\Delta z}{1 + T(z)/k}$$

Thus we have:

$$\left[1 + \frac{T(z)}{k}\right]\Delta T = g\,\mu_0\,\Delta z$$

Integrating this, we get:

$$T(z) + \frac{1}{2k}[T(z)]^2 = g\,\mu_0\,z \qquad (2)$$

Now, for $z = L$, we have $T(L) = Mg = L_0\,\mu_0\,g$. Substituting this condition in Eq. (2) gives us the value of the constant k:

$$k = \frac{g \mu_0 L_0^2}{2(L - L_0)}$$

Using this value of k, Eq. (2) can be rewritten as:

$$z = \frac{T(z)}{g \mu_0} + \frac{(L - L_0)}{g \mu_0 L_0^2} \times \frac{[T(z)]^2}{g \mu_0} \quad (3)$$

Finally, with the help of Eq. (1), we can convert Eq. (3) into a relation giving the height of the nth turn above the bottom of the Slinky as a function of n. Since the total mass M of the Slinky is equal to $L_0 \mu_0$, Eq. (1) can be rewritten:

$$T(z) = \frac{g \mu_0 L_0}{N} n(z)$$

Substituting this expression for $T(z)$ in Eq. (3) leads at once to the result:

$$z_n = \frac{L_0}{N} n + \frac{(L - L_0)}{N^2} n^2 \quad (4)$$

[This equation, with appropriate changes of symbols and coordinates, corresponds to Eq. (7) of Ref. 1.]

Thus, if our assumption that Hooke's law applies to the extended Slinky is valid, the graph of z_n vs n should be a parabola. Also, if we consider sections of the Slinky between the turns numbered n and $n + \Delta n$, where Δn is some fixed number (e.g., 5), the graph of $\Delta z/\Delta n$ vs n (where $\Delta z = z_{n+\Delta n} - z_n$) should be a straight line:

$$\frac{\Delta z}{\Delta n} = \frac{L_0}{N} + \frac{2(L - L_0)}{N^2} \left(n + \frac{\Delta n}{2}\right) \quad (5)$$

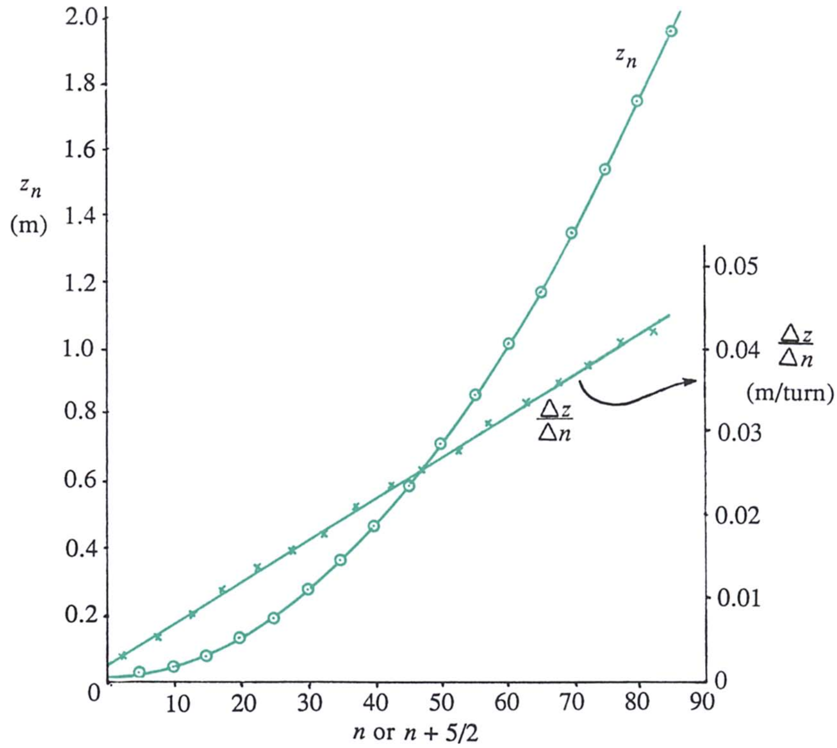

Fig. 2. Experimental graphs of z_n vs n and of $\Delta z/\Delta n$ vs $n + \Delta n/2$ (with $\Delta n = 5$ in our case). These graphs correspond to Eqs. (4) and (5) in the text.

Experiments

Observations were made on a standard Slinky. It was not convenient to use quite its whole length; measurements were made on a total of 85 turns, which when suspended vertically extended to a total length of 1.95 m. The unextended length, with the Slinky placed horizontally on a table, was about 0.10 m. (But this value is somewhat history-dependent.) Figure 2 shows graphs of z_n vs n and $\Delta z/\Delta n$ vs $n + \Delta n/2$ (for $\Delta n = 5$). The calculated slope of the latter line is $2(L - L_0)/N^2$, which is equal to 0.00051 m/turns^2; the experimental value is 0.00050 m/turns^2.

It is really striking that the Slinky, in extension, obeys Hooke's law so well, conforming remarkably closely to the predictions of Eqs. (4) and (5). Whereas most springs deviate seriously from the Hooke's-law relation for strains ($\Delta L/L_0$) of less than unity (perhaps much less), the sections of the Slinky near its top end in this experiment had strains of the order of 20; that is, the separation of successive turns was of the order of 20 times the separation in the relaxed state. This linearity is all the more surprising when one considers that, in compression, the Slinky is very highly nonlinear and cannot be compressed to much less than its relaxed length.

The observations reported here are of course very easy to do, but they provide a rather nice and unusual exercise in static equilibrium, especially if the data are obtained in the absence of any prior knowledge of the theory and the student is left with the challenge of analyzing and interpreting the results. One final remark: the center of gravity of the Slinky is of course located at the turn $n = N/2$. Students can measure the corresponding value of z, and (if they are ambitious) can compare this to what would be predicted by the theoretical formulas.

(It seems unlikely that this exercise has not already been invented by others. I would be grateful for any information about this.)

Reference

1. Thomas C. Heard and Neal D. Newby, Jr., *Am. J. Phys.* **45**, 1102–1106 (1977).